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LETTER TO THE EDITOR

A strange recursion operator demystified

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Abstract
We show that a new integrable two-component system of KdV type studied by Karasu (Kalkanlı) *et al* (2004 *Acta Appl. Math.* **83** 85–94) is bi-Hamiltonian, and its recursion operator, which has a highly unusual structure of nonlocal terms, can be written as a ratio of two compatible Hamiltonian operators found by us. Using this we prove that the system in question possesses an infinite hierarchy of *local* commuting generalized symmetries and conserved quantities in involution, and the evolution systems corresponding to these symmetries are bi-Hamiltonian as well. We also show that upon introduction of suitable nonlocal variables the nonlocal terms of the recursion operator under study can be written in the usual form, with the integration operator D_x^{-1} appearing in each term at most once.

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Using the Panilevé test, Karasu (Kalkanlı) [1] and Sakovich [2] found a new integrable evolution system of KdV type,

$$\begin{aligned} u_t &= 4u_{xxx} - v_{xxx} - 12uu_x + vu_x + 2uv_x, \\ v_t &= 9u_{xxx} - 2v_{xxx} - 12vu_x - 6uv_x + 4vv_x, \end{aligned} \tag{1}$$

and a zero-curvature representation for it [2]. Note that this system is, up to a linear transformation of u and v, equivalent to the system (16) from Foursov's [3] list of two-component evolution systems of KdV-type possessing (homogeneous) symmetries of order $k, 4 \leqslant k \leqslant 9$.

Karasu (Kalkanlı), Karasu and Sakovich [4] found that (1) has a recursion operator of the form

$$\mathfrak{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

where

$R_{11} = 3D_x^2 - 6u - 3u_x D_x^{-1}$,

$R_{12} = \big[-2D_x^5 + (2u+3v)D_x^3 + (8v_x - 4u_x)D_x^2 + (7v_{xx} - 6u_{xx} + 4u^2 - 6uv)D_x - 2u_{xxx}$
$\quad + 2v_{xxx} + 6uu_x - 3vu_x - 4uv_x + u_x D_x^{-1} \circ v_x\big] \circ \big(3D_x^3 - 4vD_x - 2v_x\big)^{-1}$,

$R_{21} = 6D_x^2 + 6u - 9v - 3v_x D_x^{-1}$,

$R_{22} = \big[-3D_x^5 + (12v - 18u)D_x^3 + (18v_x - 27u_x)D_x^2 + (14v_{xx} - 21u_{xx}$
$\quad + 12(u^2 + uv) - 9v^2)D_x - 6u_{xxx} + 4v_{xxx} + 12uu_x + 6(vu_x + uv_x)$
$\quad - 9vv_x + v_x D_x^{-1} \circ v_x\big] \circ \big(3D_x^3 - 4vD_x - 2v_x\big)^{-1}$.

Here D_x is the operator of total x-derivative:

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + \sum_{j=2}^{\infty}\left(u_{jx}\frac{\partial}{\partial u_{(j-1)x}} + v_{jx}\frac{\partial}{\partial v_{(j-1)x}}\right),$$

$u_{kx} \equiv \partial^k u/\partial x^k$, and $v_{kx} \equiv \partial^k v/\partial x^k$, see e.g. [5, 6] for further details.

Define also the variational derivatives with respect to u and v [5, 6]:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{j=1}^{\infty}(-D_x)^j\frac{\partial}{\partial u_{jx}}, \qquad \frac{\delta}{\delta v} = \frac{\partial}{\partial v} + \sum_{j=1}^{\infty}(-D_x)^j\frac{\partial}{\partial v_{jx}}.$$

We shall use the notation $\boldsymbol{u} = (u, v)^T$, $\boldsymbol{u}_{jx} = (u_{jx}, v_{jx})^T$ and $\delta/\delta\boldsymbol{u} = (\delta/\delta u, \delta/\delta v)^T$. Here and below the superscript T denotes the matrix transposition. Recall that a function that depends on $x, t, \boldsymbol{u}$ and a *finite* number of $\boldsymbol{u}_{jx}$ is said to be *local*, see e.g. [5, 6].

Because of the nonstandard structure of nonlocal terms in $\mathfrak{R}$ the known 'direct' methods (see e.g. [7–10] and references therein) for proving the locality of symmetries generated by $\mathfrak{R}$ are not applicable, so the question of whether (1) has an infinite hierarchy of local commuting symmetries remained open for a while. It was also unknown whether (1) is a bi-Hamiltonian system.

We have [4] $\mathfrak{R} = \mathfrak{M} \circ \mathfrak{N}^{-1}$, where $\mathfrak{M}$ and $\mathfrak{N}$ are some (non-Hamiltonian) differential operators of orders 5 and 3. Inspired by this fact, we undertook a search of Hamiltonian operators of orders 3 and 5 for (1), and it turned out that such operators do exist and (1) is bi-Hamiltonian. Namely, the following assertion holds.

Proposition 1. *The system (1) is bi-Hamiltonian:*

$$\boldsymbol{u}_t = \mathfrak{P}_1\delta H_0/\delta\boldsymbol{u} = \mathfrak{P}_0\delta H_1/\delta\boldsymbol{u},$$

where $H_0 = -3u + v/2$, $H_1 = 2u^2 - uv + v^2/9$, *and* $\mathfrak{P}_0$ *and* $\mathfrak{P}_1$ *are compatible Hamiltonian operators of the form*

$$\mathfrak{P}_0 = \begin{pmatrix} D_x^3 - 2uD_x - u_x & 0 \\ 0 & -9D_x^3 + 12vD_x + 6v_x \end{pmatrix}, \qquad \mathfrak{P}_1 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where

$P_{11} = D_x^5 - 4uD_x^3 - 6u_x D_x^2 + 4(u^2 - u_{xx})D_x - u_{xxx} + 4uu_x - u_x D_x^{-1} \circ u_x$,

$P_{12} = 2D_x^5 - (2u+3v)D_x^3 + 4(u_x - 2v_x)D_x^2 + (6u_{xx} - 7v_{xx} - 4u^2 + 6uv)D_x$
$\quad + 2u_{xxx} - 2v_{xxx} - 6uu_x + 3vu_x + 4uv_x - u_x D_x^{-1} \circ v_x$,

$P_{21} = 2D_x^5 - (2u+3v)D_x^3 - (10u_x + v_x)D_x^2 + (-4u^2 + 6uv - 8u_{xx})D_x$
$\quad - 2u_{xxx} - 2uu_x + 3vu_x + 2uv_x - v_x D_x^{-1} \circ u_x$,

$P_{22} = 3D_x^5 + (18u - 12v)D_x^3 + (27u_x - 18v_x)D_x^2 + (21u_{xx} - 14v_{xx} - 12u^2 - 12uv + 9v^2)D_x$
$\quad + 6u_{xxx} - 4v_{xxx} - 12uu_x - 6vu_x - 6uv_x + 9vv_x - v_x D_x^{-1} \circ v_x$.

Moreover, we have $\mathfrak{R} = 3\mathfrak{P}_1 \circ \mathfrak{P}_0^{-1}$, *and hence* $\mathfrak{R}$ *is hereditary.*

Now we are ready to prove that (1) has infinitely many local commuting symmetries.

Proposition 2. *Define the quantities* $\boldsymbol{Q}_j$ *and* H_j *recursively by the formula* $\boldsymbol{Q}_j = \mathfrak{P}_1\delta H_j/\delta\boldsymbol{u} = \mathfrak{P}_0\delta H_{j+1}/\delta\boldsymbol{u}$, $j = 0, 1, 2, \ldots$, *where* $H_0, H_1, \mathfrak{P}_0$ *and* $\mathfrak{P}_1$ *are given in proposition 1. Then* H_j, $j = 2, 3, \ldots$, *are local functions that can be chosen to be independent of x and t, and* $\boldsymbol{Q}_j$, $j = 0, 1, 2, \ldots$, *are local commuting generalized symmetries for (1).*

Thus, the evolution systems $\boldsymbol{u}_{t_j} = \boldsymbol{Q}_j$, $j = 0, 1, 2, \ldots$, are bi-Hamiltonian with respect to $\mathfrak{P}_0$ and $\mathfrak{P}_1$, and $\mathcal{H}_j = \int H_j\,\mathrm{d}x$ are in involution with respect to the Poisson brackets associated with $\mathfrak{P}_0$ and $\mathfrak{P}_1$ for all $j = 0, 1, 2\ldots$, so $\mathcal{H}_j$ are common conserved quantities for all evolution systems $\boldsymbol{u}_{t_k} = \boldsymbol{Q}_k$, $k = 0, 1, 2, \ldots$.

Proof. Let us use induction on j. Assume that $\boldsymbol{Q}_j = \mathfrak{P}_1\delta H_j/\delta\boldsymbol{u}$ is local and there exists a local function H_{j+1} such that $\boldsymbol{Q}_j = \mathfrak{P}_0\delta H_{j+1}/\delta\boldsymbol{u}$ and $\partial H_{j+1}/\partial x = \partial H_{j+1}/\partial t = 0$, and let us show that then $\boldsymbol{Q}_{j+1} = \mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}$ is local too and there exists a local function H_{j+2} such that $\boldsymbol{Q}_{j+1} = \mathfrak{P}_0\delta H_{j+2}/\delta\boldsymbol{u}$ and $\partial H_{j+2}/\partial x = \partial H_{j+2}/\partial t = 0$.

The only possibly nonlocal term in $\boldsymbol{Q}_{j+1} = \mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}$ is $-\boldsymbol{u}_x D_x^{-1}(u_x\delta H_{j+1}/\delta u + v_x\delta H_{j+1}/\delta v)$. But since H_{j+1} is independent of x by assumption, we have

$$u_x\frac{\delta H_{j+1}}{\delta u} + v_x\frac{\delta H_{j+1}}{\delta v} \equiv \boldsymbol{u}_x\cdot\frac{\delta H_{j+1}}{\delta\boldsymbol{u}} = D_x\left(H_{j+1} - \sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\boldsymbol{u}_{(m-k)x}\cdot(-D_x)^k\left(\frac{\partial H_{j+1}}{\partial\boldsymbol{u}_{mx}}\right)\right), \qquad (2)$$

where '·' stands for the scalar product of two vectors. Note that the sum in (2) is actually finite, as H_{j+1} is local by assumption. The kernel of D_x in the space of local functions is exhausted by the functions of t alone [5], so we have $D_x^{-1}(\boldsymbol{u}_x\cdot\delta H_{j+1}/\delta\boldsymbol{u}) = c_j(t) + H_{j+1} - \sum_{m=1}^{\infty}\sum_{k=0}^{m-1}\boldsymbol{u}_{(m-k)x}\cdot(-D_x)^k(\partial H_{j+1}/\partial\boldsymbol{u}_{mx})$, where $c_j(t)$ is an arbitrary function of t. Thus, $D_x^{-1}(\boldsymbol{u}_x\cdot\delta H_{j+1}/\delta\boldsymbol{u})$ is local, and so is $\boldsymbol{Q}_{j+1} = \mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}$.

Next, as $\boldsymbol{Q}_{j+1} = \mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}$, $\boldsymbol{Q}_j = \mathfrak{P}_0\delta H_{j+1}/\delta\boldsymbol{u}$ and $\mathfrak{R} = 3\mathfrak{P}_1\circ\mathfrak{P}_0^{-1}$, we can (formally) write $\boldsymbol{Q}_{j+1} = (1/3)\mathfrak{R}(\boldsymbol{Q}_j)$, cf e.g. section 7.3 of [5]. As $\mathfrak{R}$ is a recursion operator for (1), its Lie derivative along $\boldsymbol{Q}_0$ vanishes: $L_{\boldsymbol{Q}_0}(\mathfrak{R}) = 0$. Moreover, by proposition 1 the operator $\mathfrak{R}$ is hereditary, so [11] $L_{\boldsymbol{Q}_{j+1}}(\mathfrak{R}) = (1/3)^{j+1}L_{\mathfrak{R}^{j+1}\boldsymbol{Q}_0}(\mathfrak{R}) = (1/3)^{j+1}\mathfrak{R}^{j+1}L_{\boldsymbol{Q}_0}(\mathfrak{R}) = 0$. Hence $L_{\boldsymbol{Q}_{j+1}}(\mathfrak{P}_0) = 3L_{\boldsymbol{Q}_{j+1}}(\mathfrak{R}^{-1}\circ\mathfrak{P}_1) = 3\mathfrak{R}^{-1}\circ L_{\boldsymbol{Q}_{j+1}}(\mathfrak{P}_1) = 3\mathfrak{R}^{-1}\circ L_{\mathfrak{P}_1\delta H_{j+1}/\delta\boldsymbol{u}}(\mathfrak{P}_1) = 0$, cf e.g. [12]. In turn, $L_{\boldsymbol{Q}_{j+1}}(\mathfrak{P}_0) = 0$ implies that there exists a local function H_{j+2} such that $\boldsymbol{Q}_{j+1} = \mathfrak{P}_0\delta H_{j+2}/\delta\boldsymbol{u}$. This is proved along the same lines as in [13] for the second Hamiltonian structure of the KdV equation.

Finally, as the coefficients of $\mathfrak{P}_0$ and $\mathfrak{P}_1$ are independent of x and t, it is immediate that we always can choose H_{j+2} so that it is independent of x and t. The commutativity of $\boldsymbol{Q}_j = (1/3)^j\mathfrak{R}^j(\boldsymbol{Q}_0)$, $j = 0, 1, 2, \ldots$, readily follows from $L_{\boldsymbol{Q}_0}(\mathfrak{R}) = 0$ and $\mathfrak{R}$ being hereditary, see e.g. theorem 3.12 of [12]. The induction on j starting from $j = 0$ completes the proof. □

For any local H such that $\partial H/\partial x = 0$ we shall set, in agreement with (2) (see e.g. [14–17] for more details on dealing with nonlocalities),

$$D_x^{-1}\left(u_x\frac{\delta H}{\delta u} + v_x\frac{\delta H}{\delta v}\right) = H - \sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\boldsymbol{u}_{(j-k)x}\cdot(-D_x)^k\left(\frac{\partial H}{\partial\boldsymbol{u}_{jx}}\right).$$

Then, for instance, the first commuting flow for (1) reads

$$u_{t_1} = 2u_{5x} - \frac{5}{9}v_{5x} - 20uu_{xxx} + \frac{50}{9}uv_{xxx} + \frac{40}{9}vu_{xxx} - \frac{10}{9}vv_{xxx} - 50u_xu_{xx} + \frac{125}{9}u_xv_{xx}$$
$$+ \frac{40}{3}v_xu_{xx} - \frac{10}{3}v_xv_{xx} - \frac{40}{3}vuu_x + \frac{20}{9}vuv_x + 40u^2u_x - \frac{80}{9}u^2v_x + \frac{5}{9}v^2u_x,$$
$$v_{t_1} = 5u_{5x} - \frac{4}{3}v_{5x} - 40uu_{xxx} + 10uv_{xxx} + \frac{10}{3}vu_{xxx} - \frac{5}{9}vv_{xxx} - 120u_xu_{xx} + 30u_xv_{xx}$$
$$+ \frac{80}{3}v_xu_{xx} - \frac{55}{9}v_xv_{xx} + \frac{160}{3}vuu_x - 20vuv_x + \frac{40}{3}u^2v_x - \frac{40}{3}v^2u_x + \frac{35}{9}v^2v_x.$$

By proposition 2 this system is bi-Hamiltonian, and indeed we can write it as

$$\boldsymbol{u}_{t_1} = \mathfrak{P}_1 \delta H_1/\delta \boldsymbol{u} = \mathfrak{P}_0 \delta H_2/\delta \boldsymbol{u},$$

where $H_2 = \frac{7}{162}v^3 - \frac{8}{3}u^3 - \frac{5}{9}v^2u + \frac{20}{9}u^2v - u_x^2 + \frac{5}{9}v_xu_x - \frac{2}{27}v_x^2$.

Following the general procedure described in [18], we can rewrite $\mathfrak{R}$ in the standard form, with D_x^{-1} appearing in each term at most once. To this end we define nonlocal variables w, y, z by means of the formulae

$$w_x = y, \qquad w_t = -\tfrac{2}{3}wv_x + 3wu_x + \tfrac{4}{3}yv - 6yu,$$
$$y_x = \tfrac{1}{3}wv, \qquad y_t = \tfrac{2}{3}yv_x - 3yu_x + \tfrac{4}{9}wv^2 - 2wvu - \tfrac{2}{3}wv_{xx} + 3wu_{xx},$$
$$z_x = 1/w^2, \qquad z_t = -\tfrac{2}{3}(-2v + 9u)/w^2.$$

Note that the variable w above is essentially the same as in [4], and we have the following factorization [4]:

$$\left(3D_x^3 - 4vD_x - 2v_x\right)^{-1} = \tfrac{1}{3}w^2 \circ D_x^{-1} \circ w^{-2} \circ D_x^{-1} \circ w^{-2} \circ D_x^{-1} \circ w^2.$$

Moreover, as

$$D_x^{-1} \circ w^{-2} = z - D_x^{-1} \circ z \circ D_x,$$

we find that

$$\left(3D_x^3 - 4vD_x - 2v_x\right)^{-1} = \tfrac{1}{6}w^2 \circ \left(z^2 \circ D_x^{-1} \circ w^2 + D_x^{-1} \circ z^2w^2 - 2z \circ D_x^{-1} \circ w^2z\right).$$

Using the above formulae we can rewrite $\mathfrak{R}$ as follows:

$$\mathfrak{R} = \begin{pmatrix} 3 & -2/3 \\ 6 & -1 \end{pmatrix} D_x^2 + \begin{pmatrix} -6u & v/9 + 2u/3 \\ 6u - 9v & 8v/3 - 6u \end{pmatrix} + \sum_{\alpha=1}^{4} \boldsymbol{K}_\alpha D_x^{-1} \circ \boldsymbol{\gamma}_\alpha.$$

Here $\boldsymbol{\gamma}_1 = (-3, 1/2)$ is a local cosymmetry for (1), $\boldsymbol{\gamma}_\alpha = (0, w^2z^{\alpha-2})$, $\alpha = 2, 3, 4$, are *nonlocal* cosymmetries for (1), see e.g. [12] for the definition of cosymmetry; $\boldsymbol{K}_1 = (u_x, v_x)^T$ is a local symmetry of (1), and $\boldsymbol{K}_\alpha$, $\alpha = 2, 3, 4$, are *nonlocal* symmetries of (1) of the form

$$\boldsymbol{K}_2 = \Bigg(\frac{w^2z^2}{27}(3v_{xxx} - 9u_{xxx} + 27uu_x - 21vu_x - 12uv_x + 5vv_x) + \frac{wyz^2}{27}(15v_{xx} - 54u_{xx}$$
$$+ 36u^2 - 30uv + 4v^2) + \frac{z}{54}(-108u_{xx} + 30v_{xx} - 99y^2zu_x + 24y^2zv_x + 72u^2$$
$$- 60uv + 8v^2) + \frac{zy}{9w}(-33u_x + 8v_x) + \left(\frac{4}{9}v_x - \frac{11}{6}u_x\right)w^{-2}, -\frac{w^2z^2}{9}(9u_{xxx}$$
$$- 3v_{xxx} - 18(u - v)u_x + 9uv_x - 4vv_x) + \frac{wyz^2}{9}(18v_{xx} - 63u_{xx} + 36u^2$$

$$- 36uv + 5v^2) + \frac{z}{18}(36v_{xx} - 126u_{xx} + y^2z(39v_x - 162u_x) + 72(u^2 - uv)$$

$$+ 10v^2) + \frac{zy}{3w}(-54u_x + 13v_x) + \left(\frac{13}{6}v_x - 9u_x\right)w^{-2}\Bigg)^T,$$

$$\boldsymbol{K}_3 = \Bigg(\frac{2w^2z}{27}(9u_{xxx} - 3v_{xxx} - 27uu_x + 21vu_x + 12uv_x - 5vv_x) + 2u_{xx} - \frac{5}{9}v_{xx}$$

$$- \frac{2wyz}{27}(15v_{xx} - 54u_{xx} + 36u^2 - 30uv + 4v^2) + \left(y^2z + \frac{y}{w}\right)\left(\frac{11}{3}u_x - \frac{8}{9}v_x\right)$$

$$- \frac{4u^2}{3} + \frac{10uv}{9} - \frac{4v^2}{27}, \frac{2w^2z}{9}(9u_{xxx} - 3v_{xxx} - 18(u - v)u_x + 9uv_x - 4vv_x)$$

$$+ 7u_{xx} - 2v_{xx} - \frac{2wyz}{9}(18v_{xx} - 63u_{xx} + 36u^2 - 36uv + 5v^2) + 18y^2zu_x$$

$$- \frac{13}{3}y^2zv_x - \frac{y}{3w}(-54u_x + 13v_x) - 4u^2 + 4uv - \frac{5}{9}v^2\Bigg)^T,$$

$$\boldsymbol{K}_4 = \Bigg(\left(\frac{1}{9}v_{xxx} - \frac{1}{3}u_{xxx} + uu_x - \frac{7}{9}vu_x - \frac{4}{9}uv_x + \frac{5}{27}vv_x\right)w^2 + \frac{wy}{27}(-54u_{xx} + 15v_{xx} + 36u^2$$

$$- 30uv + 4v^2) + \frac{y^2}{18}(-33u_x + 8v_x), \Bigg(-u_{xxx} + \frac{1}{3}v_{xxx} + 2uu_x - 2vu_x - uv_x$$

$$+ \frac{4}{9}vv_x\Bigg)w^2 + \frac{wy}{9}(18v_{xx} - 63u_{xx} + 36(u^2 - uv) + 5v^2) + \frac{y^2}{6}(-54u_x + 13v_x)\Bigg)^T.$$

It would be interesting to investigate the properties of nonlocal symmetries $\boldsymbol{Q}_{\alpha,j} \equiv \mathfrak{R}^j(\boldsymbol{K}_\alpha)$, $j = 1, 2, \ldots$, $\alpha = 2, 3, 4$, and in particular to find out whether the commutators of $\boldsymbol{Q}_{\alpha,j}$ with local symmetries $\boldsymbol{Q}_k$ from proposition 2 yield any new symmetries for (1). We intend to address these and related issues elsewhere.

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